Morse index and causal continuity. A criterion for topology change in quantum gravity.

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Abstract

Studies in $1+1$ dimensions suggest that causally discontinuous topology changing spacetimes are suppressed in quantum gravity. Borde and Sorkin have conjectured that causal discontinuities are associated precisely with index 1 or $n-1$ Morse points in topology changing spacetimes built from Morse functions. We establish a weaker form of this conjecture. Namely, if a Morse function f on a compact cobordism has critical points of index 1 or $n-1$, then all the Morse geometries associated with f are causally discontinuous, while if f has no critical points of index 1 or $n-1$, then there exist associated Morse geometries which are causally continuous.

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1 Introduction

Within a gravitational Sum Over Histories (SOH), it is useful to obtain selection rules to eliminate certain topological transitions, based on general physical grounds, for example [1]. A reasonable criterion is that the quantum field propagation on the spacetime be finite, and Sorkin has conjectured [2, 3, 4] that this can only occur when the spacetime is causally continuous. It is therefore of interest to determine whether a given topological transition admits a causally continuous history or not. Guided by elements in surgery theory, it was further conjectured by Borde and Sorkin [4] that causal continuity of a spacetime constructed from a Morse function depends on the indices of the critical points of that Morse function . In this paper, we present a proof of a weaker form of this second conjecture, which nevertheless can be used to determine an appropriate selection rule for topology change.

There are no compelling reasons to exclude topology change from quantum gravity. Indeed, our experience of the inconsistency of the relativistic quantum mechanics of a fixed number of particles and the potential presence in quantum gravity of states that correspond to particles (topological geons) suggest that topology change *must* be included. In the SOH approach to quantum gravity [5, 6, 7, 1, 8, 9] topology change seems very naturally accommodated. Moreover we can reconcile topology change and causality, evading Geroch's theorem [10] by letting the Lorentzian metric be degenerate at a finite number of isolated singularities. This was the original motivation to consider Morse spacetimes, which depart only mildly from being globally Lorentzian, and can be defined in all possible cobordisms.

Let $(\mathcal{M}, V_0, V_1)$ be a compact n-dimensional cobordism, with $\partial\mathcal{M} = V_0 \amalg V_1$, h be a Riemannian metric on $\mathcal{M}$ and $f : \mathcal{M} \to [0,1]$ a Morse function, with $f^{-1}(0) = V_0$, $f^{-1}(1) = V_1$, which has r critical points, $\{p_k\}$, in the interior of $\mathcal{M}$. The critical points of f are those where $\partial_\mu f = 0 \; \forall\mu$. What characterises a Morse function is that the Hessian $\partial_\mu\partial_\nu f$ of f is an invertible matrix at each of its critical points, often called Morse points. The Morse index λ_k of the critical point p_k is the number of negative eigenvalues of the matrix $(\partial_\mu\partial_\nu f)(p_k)$. From h, f and an arbitrary real constant greater than 1, denoted ζ, we can define a *Morse metric* in $\mathcal{M}$:

$$g_{\mu\nu} = (h^{\rho\lambda}\partial_\rho f \partial_\lambda f) h_{\mu\nu} - \zeta \partial_\mu f \partial_\nu f, \tag{1}$$

This metric is Lorentzian everywhere, except at the critical points of f. The *Morse geometry* (M, g) associated with $\mathcal{M}$, h, f and ζ is the globally Lorentzian spacetime induced in $M = \mathcal{M} - \{p_k\}$, the manifold that remains after excising the critical points from $\mathcal{M}$.[1] We will adopt the notation that calligraphic capital letters refer to the unpunctured manifolds and their Roman counterparts to the corresponding punctured

[1]The term "geometry" here does not imply quotienting by the action of any group of diffeomorphisms. We use it to distinguish the current case from future work in which the Morse point will regain its status as a physically present point and the pair $(\mathcal{M}, g)$ will be known as a *Morse spacetime*.

manifolds. A general cobordism $\mathcal{M}$ can be decomposed into a finite sequence of elementary cobordisms $\mathcal{E}_k$, each containing one critical point. If $\mathcal{E}$ is an elementary cobordism, we will refer to a Morse geometry (E, g), defined in $E = \mathcal{E} - p$, as an elementary Morse geometry. The vector field $h^{\mu\nu}\partial_\nu f$ is time-like with respect to the metric g, so that we can think of f as a time-function on (M, g). We refer the reader to [4, 2, 3] for a more expository account.

As detailed in [2], surgery theory suggests that the index of f at its critical points determines the "continuity" in the causal structure of a Morse geometry. For example, when the index is 1 disconnected regions of space seem suddenly to come into contact with one another. Borde and Sorkin put forward the following conjecture:

Conjecture 1 **[Borde-Sorkin conjecture]** *Given a compact cobordism $\mathcal{M}$ and a Morse function $f : \mathcal{M} \to \mathbb{R}$ with critical points $\{p_k\}$, then a Morse geometry (M, g) defined through f, is causally continuous if and only if none of the points p_k has Morse index 1 or $n-1$.*

In [4], we studied a particular class of Morse geometries defined in the neighbourhood of a critical point. From Morse theory we know that around a critical point p of a Morse function f in an n-dimensional cobordism, there is a round neighbourhood $\mathcal{D}_\epsilon$, of radius ϵ, in local coordinates $(x_1, \cdots, x_\lambda, y_1, \cdots y_{n-\lambda})$, in which the Morse function takes the canonical form [11]:

$$f = c - \sum_{i=1}^{\lambda} x_i^2 + \sum_{j=1}^{n-\lambda} y_j^2. \qquad (2)$$

$c = f(p)$ is the critical value and λ is the Morse index of p. The neighbourhood Morse geometry we studied was defined on a particular neighbourhood of p in $\mathcal{D}_\epsilon$ (called Q_δ and chosen so that no spurious causal discontinuity could arise from the boundaries). The metric was constructed from f and the Cartesian flat Riemannian metric with interval $ds_h^2 = \sum_{i=1}^{\lambda} dx_i^2 + \sum_{j=1}^{n-\lambda} dy_j^2$. We verified conjecture 1 for these "Cartesian" neighbourhood Morse geometries.

For a full proof of the conjecture, our earlier result needs to be generalised in two main directions. First we have to verify that the conjecture holds in neighbourhood Morse geometries constructed from arbitrary Riemannian metrics. Secondly we need to embed the neighbourhood Morse geometries in the original Morse geometry and show that the latter is causally continuous if and only if the neighbourhoods are. In this work we present progress made along these two lines.

After a section containing background material, we start by investigating the general Morse geometry in the neighbourhood of a Morse point p. In section 3 we use results from dynamical systems to establish certain topological properties of the flow

of the timelike vector field $\xi^\mu = h^{\mu\nu}\partial_\nu f$. This helps us construct past and future sets that can be identified with the chronology of the critical point.

In section 4 we show that that any Morse geometry (M, g) constructed from a Morse function which has an index 1 or $n-1$ point is causally discontinuous.

In section 5 we consider cobordisms $\mathcal{M}$ with Morse functions f with no index 1 and $n-1$ points. Here, we show that one can always construct a causally continuous Morse geometry built from f. We prove this in several steps. First, we establish that a Morse geometry (M, g) is causally continuous if and only if all the elementary Morse geometries which stacked together give M are causally continuous. Then we show that an elementary Morse geometry (E, g) with a neighbourhood Morse geometry (N, g) embedded in it, is causally continuous if and only if (N, g) is. Finally we show that we can always choose an appropriate Riemannian metric which guarantees that there is a causally continuous neighbourhood Morse geometry around each Morse point.

In section 6 we summarise our results by stating them together as a proposition which is a weaker version of the Borde-Sorkin Conjecture.

2 Review of Causal Structure

We review some relevant material on the causal structure of globally Lorentzian spacetimes [12, 13, 14, 15]. We also include a few simple extensions of the usual results that will be needed in later sections. All spacetimes in this paper are assumed to be time-orientable and distinguishing and if a spacetime includes boundaries they are restricted to be spacelike initial and final boundaries.

A timelike curve in a spacetime (M, g) is a differentiable curve $\gamma : \mathbb{R} \to M$ whose tangent vector is everywhere timelike. A causal curve is one whose tangent vector is timelike or null. We say that a timelike(causal) curve starting at $x \in M$ is future-inextendible if it has no future endpoint in M except possibly on the final boundary of M. One defines past-inextendible timelike(causal) curves similarly. Given a pair of points x, y in M, we write $x << y$ if there exists a future-directed timelike curve from x to y, and $x < y$ if there exists a future-directed causal curve from x to y. The chronological future of a point $x \in M$ is $I^+(x) = \{y : \ x << y\}$, while its causal future is $J^+(x) = \{y : \ x < y\}$. The chronological and causal pasts are defined dually.

Clearly, $I^\pm(x) \subset J^\pm(x)$. We also have (i) $I^\pm(S)$ is open for every set $S \subset M$, (ii) $J^+(x) \subset \overline{I^+}(x)$ for every point in the interior of M, (iii) $x < y$ and $y << z \Rightarrow x << y$ and (iv) $y \in \overline{I^+}(x)$ and $z \in \overline{I^+}(y) \Rightarrow z \in \overline{I^+}(x)$. Again, the dual statements hold. For $U \subseteq M$, we write $I^+(x, U)$ to denote the chronological future of x in the spacetime $(U, g_{|U})$. The sets $I^-(x, U)$ and $J^\pm(x, U)$ are defined similarly. An open subset U

of a globally Lorentzian spacetime (M, g) is said to be I-convex in M if for any pair of points x,$y \in U$ we have $I^+(x) \cap I^-(y) \subset U$. It can be readily verified that if U is I-convex, then for any $x \in U$, $I^{\pm}(x, U) = I^{\pm}(x) \cap U$ and that the interval $I^+(V) \cap I^-(W)$ between any two subsets V, W of M is I-convex.

The common past of an open set S is $\downarrow S \equiv \mathrm{Int}\,(\{q : \;\; q << s \;\forall s \in S\})$, its common future $\uparrow S$ is defined dually. It is immediate that $I^-(x) \subset \downarrow I^+(x)$ and $I^+(x) \subset \uparrow I^-(x)$. If U is a subspacetime of M then the common past relative to U of open set $S \subset U$ is written $\downarrow_U S \equiv \mathrm{Int}\,(\{q : q \in I^-(s, U),\; \forall s \in S\})$.

Definition 1 A spacetime (M, g) is causally continuous when either of the following equivalent conditions holds:

(A) $I^-(x) = \downarrow I^+(x)$ and $I^+(x) = \uparrow I^-(x)$ for every point x in $M - \partial M$.

(B) $x \in \overline{I^-(y)} \Leftrightarrow y \in \overline{I^+(x)}$ for every pair of points $x, y \in M$.

These are two of the six characterisations of causal continuity given by Hawking and Sachs [13], with a difference: Hawking and Sachs enforce (A) in the whole of M, which is assumed boundaryless. A Morse geometry M contains spacelike boundaries V_0 and V_1, where the condition (A) trivially fails and hence the need for a modification. Condition (B), however, can be extended to the boundary without difficulty.

We also need the definitions of causality, strong causality, stable causality and global hyperbolicity. A spacetime (M, g) is *causal* if there are no closed causal curves in M. A spacetime (M, g) is *strongly causal* if every point in M has neighbourhoods that no causal curve intersects more than once. A spacetime (M, g) is *stably causal* if there exists a spacetime (M, g') such that the lightcones of g' are everywhere wider than those of g and which is causal. Stable causality is equivalent [12] to the existence of a global time function on M. A spacetime is *globally hyperbolic* if it contains a spacelike hypersurface which every inextendible causal curve intersects at exactly one point. The conditions listed are arranged from stronger to weaker in the chain of implications: global hyperbolicity $\Rightarrow$ causal continuity $\Rightarrow$ stable causality $\Rightarrow$ strong causality $\Rightarrow$ causality.

Given a closed set V of a spacetime (M, g) the *future domain of dependence* of V, denoted $D^+(V)$, is the set of all points $x \in M$ such that every past-inextendible causal curve through x intersects V. Similarly one defines the past domain of dependence $D^-(V)$. Their union, $D(V) = D^+(V) \cup D^-(V)$, is the *domain of dependence* of V. It consists of all points $x \in M$ such that every inextendible causal curve through x intersects V. Clearly $V \subset D(V)$. For any closed $V \subset M$, the set $\mathrm{Int}\,(D(V))$ is I-convex. A set V in M is said to be achronal if no two points in V are chronologically related. It is shown in [12] that if V is a closed achronal set in M then $\mathrm{Int}\,(D(V))$ is globally hyperbolic and therefore causally continuous.

We finally note the special properties of Morse geometries (M, g) which, with their Morse points excised, are globally Lorentzian. Any Morse geometry (M, g) is stably causal, since the Morse function f on M is a global time function. In fact, causal continuity is the weakest of the conditions listed above which is not guaranteed hold in all Morse geometries, as we will see in the next few sections.

3 Dynamical systems in a Morse neighbourhood

As mentioned in the introduction, in [4] conjecture 1 was verified for spacetimes defined in a certain punctured neighbourhood Q of a critical point, with metric constructed from the Cartesian Riemannian metric $\delta_{\mu\nu}$. These neighbourhood Morse geometries are to be regarded as embedded in a Morse geometry (M, g). If we start from a general (M, g), the Riemannian metric h from which g is constructed will usually be different from $\delta_{\mu\nu}$ in any neighbourhood of a Morse point. In this section we investigate the causal structure in these more general neighbourhood Morse geometries using some results from dynamical systems.

Consider the neighbourhood Morse geometry (D_ϵ, g) around p of index λ in which the Morse function takes the form $f = c - \sum_1^\lambda (x^i)^2 + \sum_1^{n-\lambda} (y^j)^2$ and $D_\epsilon = \{(x^i, y^j) : 0 < \sum_1^\lambda (x^i)^2 + \sum_1^{n-\lambda} (y^j)^2 < \epsilon^2\}$. We place no restrictions on the Riemannian metric $h_{\mu\nu}$ in equation (1). In what follows it will sometimes be convenient to group together the coordinates (x^i, y^j) in a single set $\{X^\mu\}$ with $X^\mu = x^\mu$ when $\mu \leq \lambda$ and $X^\mu = y^{\mu-\lambda}$ when $\mu > \lambda$.

In our earlier investigations of neighbourhood Morse geometries with $h_{\mu\nu} = \delta_{\mu\nu}$ [4], we defined certain past and future sets, $\mathcal{P}$ and $\mathcal{F}$, which could be interpreted as the chronological past and future of the critical point. These played a crucial role in the analysis, and so our first step is to construct $\mathcal{P}$ and $\mathcal{F}$ for arbitrary $h_{\mu\nu}$.

We start by looking at the vector field $\eta^\mu = \delta^{\mu\nu}\partial_\nu f$, which is a gradient-like vector field associated with f. This means that η satisfies the conditions (i) η^μ is transverse to f, $\eta^\mu \partial_\mu f > 0$, and (ii) in some neighbourhood of the Morse point , $\mathcal{D}_\epsilon$, η^μ takes the canonical form, $\eta^\mu = (-x_1 \cdots - x_\lambda, y_1 \cdots y_{n-\lambda})$, in local coordinates in which the Morse function takes *its* canonical form [11]. Figure 1 shows some representative integral curves of η in two dimensions, for an index 1 critical point, which are hyperbolae. Morse theory tells us, further, that the integral submanifolds of η through p (referred to as "basins" in the language of dynamical systems) are the discs $\mathcal{D}_\epsilon^\lambda$ and $\mathcal{D}_\epsilon^{n-\lambda}$ constructed from integral curves which end and begin at the critical point, respectively,

$$\begin{aligned}
\mathcal{D}_\epsilon^\lambda &= \left\{(x^1, \ldots, x^\lambda, 0, \ldots, 0) : \sum {x^i}^2 < \epsilon\right\} \\
\mathcal{D}_\epsilon^{n-\lambda} &= \left\{(0, \ldots, 0, y^1, \ldots, y^{n-\lambda}) : \sum {y^j}^2 < \epsilon\right\}
\end{aligned}$$

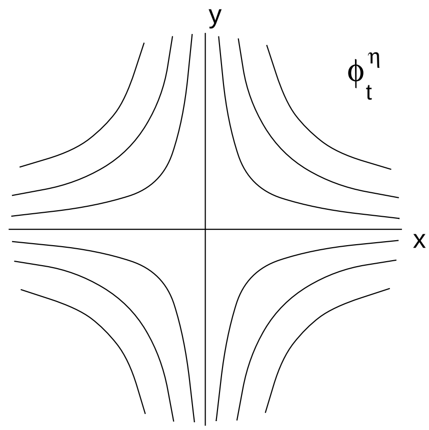

Figure 1: Integral flow of the vector field $\eta = (-2x, 2y)$, the gradient-like vector field for the Morse function $f = -x^2 + y^2$ in the 2-dimensional disc.

Consider also the vector field, $\xi^\mu = h^{\mu\nu}\partial_\nu f$ for the neighbourhood Morse spacetime (D_ϵ, g), which is manifestly time-like with respect to g. When $g = \delta$, as in [4], $\xi = \eta$, but in general they are different and η will not necessarily be timelike. Since ξ is timelike, the properties of its flow are important to the causal structure. We will now relate it to the flow of η which we know exactly. Indeed, we will show that the flows of ξ and η are topologically equivalent, in the sense that the integral curves of one are mapped homeomorphically onto the integral curves of the other. In particular, this implies that the basins of ξ swept out by the timelike integral curves are topologically D^λ and $D^{n-\lambda}$. This is exactly what we need to construct $\mathcal{P}$ and $\mathcal{F}$.

We begin the proof of the topological equivalence of ξ and η by first introducing certain properties of dynamical systems.

A *dynamical system* in $\mathbb{R}^n$ is given by a system of equations $\dot{X}^\mu = \xi^\mu(X)$, where X^μ are the Cartesian coordinates in $\mathbb{R}^n$ and each ξ^μ a differentiable real function on $\mathbb{R}^n$. Solving this system amounts to finding its integral curves. That is, for each point $X_0 \in \mathbb{R}^n$, we seek the unique curve $X(t)$ which satisfies, (i) $X(0) = X_0$ and (ii) $\dot{X}^\mu(t) = \xi^\mu(X(t))$.

The *flow* of ξ, which we denote by ϕ_t^ξ, is the one parameter family of local diffeomorphisms defined by pushing points along the integral curves of ξ. More specifically, in a neighbourhood of each point x and for each t in some finite range $(-\epsilon, \epsilon)$, the map $y \to \phi_t^\xi(y)$ is a diffeomorphism. A point p is a fixed point of the dynamical system if $\xi^\mu(p) = 0$ for every μ.

A *linear system*, with fixed point at the origin, is defined by the equations, $\dot{X}^\mu = A^\mu_\nu X^\nu$, where A^μ_ν is some constant matrix. An arbitrary dynamical system ξ^μ admits a linear approximation around an isolated fixed point p which we denote $\tilde{\xi}^\mu$ and is

given by $\dot{X}^\mu = \tilde{\xi}^\mu = (\partial_\nu \xi^\mu)(p) X^\nu$.

The integral flows ϕ_t^ξ and ϕ_t^η of two vector fields ξ, η in $\mathbb{R}^n$ are said to be *topologically equivalent* if there exists a bijection $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ taking the flow ϕ_t^ξ to the flow ϕ_t^η, so that $\Psi \circ \phi_t^\xi = \phi_t^\eta \circ \Psi$ for any $t \in \mathbb{R}$. The two topologically equivalent flows are thus related by the change of coordinates Ψ.

We now state two important theorems on dynamical systems, given for example in [16, 17].

Theorem 1 *Two linear systems having no eigenvalues with real part zero are topologically equivalent if and only if the number of eigenvalues with negative real part is the same for the two systems.*

Theorem 2 *Let ϕ_t^ξ be the integral flow of a dynamical system ξ^μ in $\mathbb{R}^n$ which has a fixed point at the origin $\vec{0}$ and $\phi_t^{\tilde{\xi}}$ the integral flow of its linear approximation. Suppose the matrix $A^\mu_\nu = \partial_\nu \xi^\mu(\vec{0})$ which determines the linear system $\tilde{\xi}$ has no purely imaginary eigenvalues. Then there is a neighbourhood U of the origin and a homeomorphism Ψ from U onto another neighbourhood V of the origin such that $\Psi \circ \phi_t^{\tilde{\xi}} = \phi_t^\xi \circ \Psi$.*

What this theorem tells us is that a non-linear system is topologically equivalent to its linearisation in a neighbourhood of the fixed point, provided the linear system has no purely imaginary eigenvalues. Clearly, the range of t involved in $\Psi \circ \phi_t^{\tilde{\xi}} = \phi_t^\xi \circ \Psi$ is restricted for each $q \in U$ so that $\phi_t^{\tilde{\xi}}(q)$ lies in U. Using these results we can now establish,

Lemma 1 *Let $\mathcal{D}_\epsilon$ be the round neighbourhood of radius ϵ about the critical point of the Morse function f and g be the Morse metric (1) constructed from f and a Riemannian metric h. Let ϕ_t^η be the flow of the gradient-like vector field $\eta^\mu = \delta^{\mu\nu}\partial_\nu f$ in $\mathcal{D}_\epsilon$ and ϕ_t^ξ that of the vector field $\xi^\mu = h^{\mu\nu}\partial_\nu f$ in $\mathcal{D}_\epsilon$, which is timelike with respect to g. Then there is a neighbourhood $\mathcal{D}_{\epsilon'}$ with $\epsilon' < \epsilon$ and a homeomorphism $\Psi : \mathcal{D}_{\epsilon'} \to \mathcal{N} \subset \mathcal{D}_\epsilon$, which is a topological equivalence between ϕ_t^η in $\mathcal{D}_{\epsilon'}$ and ϕ_t^ξ in $\mathcal{N} \equiv \Psi(\mathcal{D}_{\epsilon'})$.*

Proof As we observed earlier, both dynamical systems have a single fixed point at p, which we take to be the origin in the coordinates $\{X^\mu\}$. Since the disc is homeomorphic to $\mathbb{R}^n$, theorems 1 and 2 can be readily applied.

The flow ϕ_t^η is associated with the linear dynamical system $\dot{X}^\mu = \eta^\mu = 2\Lambda^\mu_\nu X^\nu$, where Λ is the diagonal matrix $(-1, \ldots, -1, 1, \ldots, 1)$, with λ negative eigenvalues

and $n-\lambda$ positive eigenvalues[2]. The flow of ϕ_t^ξ is determined by the dynamical system $\dot{X}^\mu = \xi^\mu = h^{\mu\nu}\partial_\nu f$, which is in general non-linear, since the metric components $h^{\mu\nu}$ are arbitrary functions of the coordinates $\{X^\mu\}$. The linearisation of this dynamical system around its fixed point p is given by:

$$\dot{X}^\mu = \partial_\rho(h^{\mu\nu}\partial_\nu f)\, X^\rho \tag{3}$$

where the partial derivatives are evaluated at p. Since here $\partial_\mu f = 0$, the derivatives of $h^{\mu\nu}$ do not appear and we are left with:

$$\begin{aligned} \dot{X}^\mu &= H^{\mu\nu}(\partial_\rho\partial_\nu f)_p\, X^\rho \\ &= 2H^{\mu\nu}\Lambda_{\nu\rho}\, X^\rho \end{aligned} \tag{4}$$

where H is the symmetric positive-definite matrix $H^{\mu\nu} = h^{\mu\nu}(p)$. Denoting the matrix that governs the linear system (4) by $A^\mu_\nu = 2H^{\mu\rho}\Lambda_{\rho\nu}$, we may now assert:

Claim 1 *The eigenvalues of A^μ_ν are all real. There are λ negative ones and $n-\lambda$ positive ones.*

Proof By symmetry and positive-definiteness, there is an orthogonal transformation that takes H into a diagonal matrix D with strictly positive entries. That is $H = ODO^T$. The matrix $H^{1/2} = OD^{1/2}O^T$ is also symmetric and positive-definite. We know that a similarity transformation preserves eigenvalues, while conjugacy[3] preserves the number of negative, positive and zero eigenvalues (this is Sylvester's theorem). Now:

$$H^{-1/2}AH^{1/2} = 2H^{1/2}\Lambda H^{1/2} \qquad \text{and} \tag{i}$$
$$H^{-1/2}(2H^{1/2}\Lambda H^{1/2})(H^{-1/2})^T = 2\Lambda \tag{ii}$$

From equation (i) we see that the eigenvalues of A are the same as those of the symmetric matrix $2H^{1/2}\Lambda H^{1/2}$, and hence they must be real. Then equation (ii) tells us that the distribution of eigenvalues of A into positive, negative and zero is the same as for the matrix 2Λ. Hence, the claim. $\square$

By claim 1 and theorem 1 we see that the flows associated with the linear systems η and $\tilde{\xi}$ are topologically equivalent in $\mathcal{D}_\epsilon$. On the other hand, combining the claim with theorem 2 we deduce that the flows of $\tilde{\xi}$ and ξ are topologically equivalent in a neighbourhood of the critical point. We can then compose the corresponding homeomorphisms to conclude that there is a neighbourhood $\mathcal{U}$ of p in $\mathcal{D}_\epsilon$ and a homeomorphism Ψ from $\mathcal{U}$ onto another neighbourhood $\mathcal{V}$ of p in $\mathcal{D}_\epsilon$ such that for any $q \in \mathcal{U}$ and t with $\phi_t^\eta(q) \in \mathcal{U}$ we have $\Psi \circ \phi_t^\eta(q) = \phi_t^\xi \circ \Psi(q)$. In figure 2 we have represented the two step construction of Ψ.

[2] We regard Λ^μ_ν as the result of raising the Hessian $\Lambda_{\mu\nu} = \frac{1}{2}\partial_\mu\partial_\nu f$ with the Cartesian flat metric $\delta^{\mu\nu}$. We raise and lower the indices on Λ with δ so that its entries have always the same value, irrespective of the index position.

[3] The conjugate of a matrix A by an invertible matrix B is the matrix BAB^T.

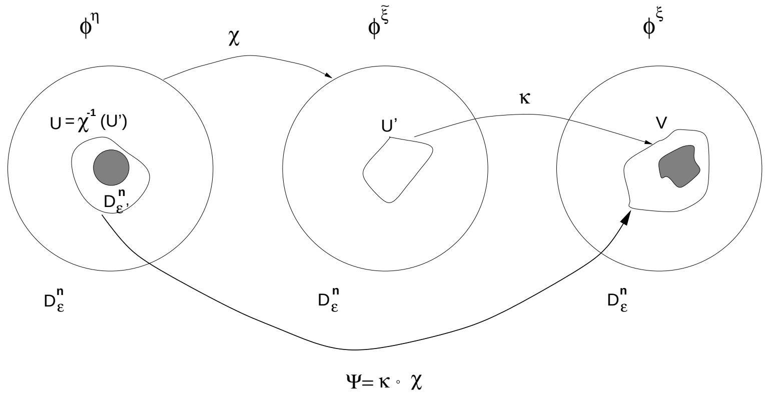

Figure 2: We compare the flows of the gradient-like vector field η^μ and the timelike vector field $\xi^\mu = h^{\mu\nu}\partial_\nu f$ using the linearisation $\tilde{\xi}^\mu$ of ξ^μ as an intermediate step. Here the homeomorphism $\chi : \mathcal{D}_\epsilon \to \mathcal{D}_\epsilon$ is a topological equivalence between the linear flows of η and $\tilde{\xi}$. The homeomorphism $\kappa : U' \to V$ gives a topological equivalence between the flows of $\tilde{\xi}$ in U' and ξ in V. We will consider the restriction of $\Psi = \kappa \circ \chi$ from $U = \xi^-1(U')$ to V, to a round neighbourhood $\mathcal{D}_{\epsilon'}$ of p, where the topological properties of the flow of η are easy to see.

The result now follows, for $\mathcal{U}$ contains a neighbourhood $\mathcal{D}_{\epsilon'}$ of p, for some $\epsilon' < \epsilon$, and the restriction $\Psi_{|\mathcal{D}_{\epsilon'}}$ maps the flow of η homeomorphically to the flow of ξ in $\mathcal{N} \equiv \Psi(\mathcal{D}_{\epsilon'})$. $\Box$

As we stated earlier, our aim in proving the topological equivalence between ξ and η is that this implies that the basins of ξ in $\mathcal{N}$ are topologically the same as those of η in $\mathcal{D}_{\epsilon'}$. We denote the "ingoing" one by $\mathcal{N}^\lambda \equiv \Psi(\mathcal{D}^\lambda_{\epsilon'})$ and the "outgoing" one by $\mathcal{N}^{n-\lambda} \equiv \Psi(\mathcal{D}^{n-\lambda}_{\epsilon'})$.

Now, let us consider the neighbourhood Morse geometry (N, g), where $N = \mathcal{N} - p$. This is a subspace of the original Morse geometry (M, g). In (N, g) the basins of ξ are replaced by the two punctured disks, $N^\lambda = \mathcal{N}^\lambda - p$ and $N^{n-\lambda} = \mathcal{N}^{n-\lambda} - p$, which are swept out by congruences of timelike curves, that respectively "end" and "begin" at p. Let us define the past set, $\mathcal{P} \equiv I^-(N^\lambda, N)$ and the future set $\mathcal{F} \equiv I^+(N^{n-\lambda}, N)$ from these punctured discs. Clearly, $\mathcal{P}$ lies in the subcritical region of N and $\mathcal{F}$ in the supercritical region. We now show that every point in $\mathcal{P}$ is to the past of every other point in $\mathcal{F}$.

Claim 2 *Any pair of points s, q with $s \in \mathcal{P}$ and $q \in \mathcal{F}$ satisfy $s \in I^-(q, N)$.*

Proof By construction, for every $s \in \mathcal{P}$ there exists an $s' \in N^\lambda$ such that $s \in I^-(s', N)$ and similarly, for every $q \in \mathcal{F}$, there exists a $q' \in N^{n-\lambda}$ such that $q \in I^+(q', N)$. Consider open neighbourhoods $U_{s'}$ of s' and $U_{q'}$ of q' with $U_{s'} \subset I^+(s, N)$ and $U_{q'} \subset I^-(q, N)$. We now show that there are points $u \in U_{s'}$ and $v \in U_{q'}$ connected

by an integral curve of the timelike vector field ξ, contained in N, which implies that $u \in I^-(v, N)$ as required.

The flow of ξ in N is mapped to the flow of η in $D_{\epsilon'} \equiv \mathcal{D}_{\epsilon'} - p$ by Ψ^{-1}. So, consider $\Psi^{-1}(s') \in D^{\lambda}_{\epsilon'} \equiv \mathcal{D}^{\lambda}_{\epsilon'} - p$ with coordinates $(\vec{x}_{s'}, \vec{0})$ and $\Psi^{-1}(q') \in D^{n-\lambda}_{\epsilon'} \equiv \mathcal{D}^{n-\lambda}_{\epsilon'} - p$ with coordinates $(\vec{0}, \vec{y}_{q'})$. Consider a point close to $(\vec{x}_{s'}, \vec{0})$, i.e., $(\vec{x}_{s'}, \delta\vec{y}_{q'})$, where $\delta > 0$ is some (small) constant. The integral curve of η through $(\vec{x}_{s'}, \delta\vec{y}_{q'})$ is given by $(e^{-2t}\vec{x}_{s'}, e^{2t}\delta\vec{y}_{q'})$. This curve passes through $(\delta\vec{x}_{s'}, \vec{y}_{q'})$ when $t = \frac{1}{2}\ln\frac{1}{\delta}$. The image of this curve under Ψ in N is a timelike curve and by continuity δ can be chosen small enough so that it passes arbitrarily close to s' and q'. $\Box$

We end this section with a couple of results for arbitrary λ. These do not use the dynamical systems technology, but will be useful in the proofs of the next few sections.

First we show that no causal curve that begins and ends close enough to p can stray too far from it.

Claim 3 *Let (U, g) be a neighbourhood Morse geometry around a Morse point p of the Morse geometry (M, g). Then there is a neighbourhood $U' \subset U$ of p such that any causal curve γ between two points x, y in U' must be contained in U.*

Proof Consider $x, y \in U$ and γ a future directed causal curve from x to y. Suppose that γ leaves U between x and y. Since γ is causal we must have $g_{\mu\nu}dx^\mu dx^\nu \leq 0$ at every point in γ. For the Morse metric $g_{\mu\nu}$ this means:

$$h^{\rho\sigma}\partial_\rho f\,\partial_\sigma f\, h_{\mu\nu}dx^\mu dx^\nu - \zeta\,\partial_\mu f\,\partial_\nu f\, dx^\mu dx^\nu \leq 0$$

$$\text{or} \qquad \zeta\,(\partial_\mu f\, dx^\mu)^2 \geq h^{\rho\sigma}\partial_\rho f\,\partial_\sigma f\, h_{\mu\nu}dx^\mu dx^\nu$$

Since γ is future-directed, $\partial_\mu f\, dx^\mu > 0$ at every point of γ so that taking square roots and then rearranging we get:

$$\zeta^{1/2}\partial_\mu f\, dx^\mu - (h^{\rho\sigma}\partial_\rho f\,\partial_\sigma f\,)^{1/2}\,(h_{\mu\nu}dx^\mu dx^\nu)^{1/2} \geq 0$$

Integrating this expression along γ we obtain:

$$\zeta^{1/2}\,(f(y) - f(x)) - \int_x^y (h^{\rho\sigma}\partial_\rho f\,\partial_\sigma f\,)^{1/2}(h_{\mu\nu}dx^\mu dx^\nu)^{1/2} \geq 0 \tag{5}$$

We now establish a lower bound for the above integral. Let $D_A \subset D_B \subset U$ be round, open n-discs centred on p. Their boundaries are the $n-1$-spheres S_A and S_B, respectively. Suppose $x, y \in D_A$, then the curve γ must have at least one arc $\gamma([t_A, t_B])$ with $t_x < t_A < t_B < t_y$ which lies entirely in the cylinder $C = \overline{D_B} - D_A$

and with $x_A = \gamma(t_A) \in S_A$ and $x_B = \gamma(t_B) \in S_B$ and another such arc coming back in from S_B to S_A. Since $\partial_\mu f$ is nowhere vanishing in the compact set C, the function $(h^{\mu\nu}\partial_\mu f \partial_\nu f)^{1/2}$ must have a minimum $\mu > 0$ in this region. It follows that,

$$\int_{x_A}^{x_B} (h^{\rho\sigma}\partial_\rho f\,\partial_\sigma f\,)^{1/2}(h_{\mu\nu}dx^\mu dx^\nu)^{1/2} \geq \mu \int_{x_A}^{x_B} (h_{\mu\nu}dx^\mu dx^\nu)^{1/2}$$

Finally, we must show that the Riemannian length of any path from S_A to S_B is larger than some fixed positive number. Now for any Riemannian metric h in a manifold M, the distance function $d : M \times M \to [0, \infty)$ defined by

$$d(x, y) = Inf_{\gamma\in\Omega(x,y)}\{\int_\gamma (h_{\mu\nu}dx^\mu dx^\nu)^{1/2}\} \tag{6}$$

where $\Omega(x, y)$ is the set of piecewise differentiable curves from x to y, is continuous and satisfies $d(x, y) = 0 \Leftrightarrow x = y$. It follows that for any two disjoint compact subsets A and B in M the number $d(A, B) = Inf\,\{\,d(x, y) : x \in A,\, y \in B\}$ is positive. Hence, since S_A and S_B are disjoint and compact, $d(S_A, S_B)$ is some positive number, d_{AB}. Thus

$$\int_x^y (h^{\rho\sigma}\partial_\rho f\,\partial_\sigma f\,)^{1/2}(h_{\mu\nu}dx^\mu dx^\nu)^{1/2} \geq 2\mu\, d_{AB} \tag{7}$$

which we can then insert in equation (5) to obtain:

$$f(y) - f(x) \geq 2\zeta^{-1/2}\,\mu\, d_{AB} \tag{8}$$

Let $\alpha = \zeta^{-1/2}\,\mu\, d_{AB}$. If we define $U' = D_A \cap f^{-1}((c-\alpha, c+\alpha))$ then any causal curve γ between two points in U' cannot leave U.□

Finally we prove the almost obvious result:

Claim 4 *Given a Morse geometry (M, g), let $N = \mathcal{N} - p$ be the punctured neighbourhood of a critical point p with $\mathcal{N}$ as in lemma 1 so that $\mathcal{P} = I^-(N^\lambda, N)$ and $\mathcal{F} = I^+(N^{n-\lambda}, N)$ are defined as before. Then any neighbourhood U of p contains points in $\partial\mathcal{P}$ and $\partial\mathcal{F}$.*

Proof Let $\tilde{U}$ be the connected component of $U \cap N$ around the critical point. Pick a point x in $V_c \cap \tilde{U}$ where $V_c \equiv \{x \in M : f(x) = c\}$ is the critical surface. The same arguments used in the proof of claim 3 guarantee that there exists a neighbourhood $U_x \subset \tilde{U}$ of x with $U_x \cap \mathcal{F} = \emptyset$. Take any curve $\gamma : [0, 1] \to U'$ with $\gamma(0) = x$ and $\gamma(1) = y \in D_\epsilon^{n-\lambda}$ and let $\tau = Sup_t\,\{\forall\, t' < t\;\gamma(t') \notin \mathcal{F}\}$. Then $\tau \in (0, 1)$, since there is a neighbourhood of y in $\mathcal{F}$, and $\gamma(\tau) \in \partial\mathcal{F}$. That U contains points in $\partial\mathcal{P}$ is proved similarly. □

4 Index $\lambda = 1, n-1$

We can now prove one half of the Borde-Sorkin conjecture.

Lemma 2 *A Morse geometry (M, g) with an index 1 or $n-1$ Morse point p is causally discontinuous.*

Proof We consider only the index 1 case, since the $n-1$ result follows by the dual argument. Pick a neighbourhood $\mathcal{N}$ of p as in lemma 1, with coordinates $\{x, y^\mu\}$ so that $f = c - x^2 + \sum_{\mu=1}^{n-1}(y^\mu)^2$. Let $\Pi = \{q \in \mathcal{N} : f(q) < c\}$ and $\Phi = \{q \in \mathcal{N} : f(q) > c\}$ be the subcritical and supercritical regions of the neighbourhood, respectively. Π comprises two disconnected components, Π_1 and Π_2, corresponding to $x > 0$ and $x < 0$, while Φ is connected. We know from lemma 1 that the basins of the timelike vector field ξ in $\mathcal{N}$ have the same topology and relative position as those of the gradient-like vector field η in $D_{\epsilon'}$. Thus, the outgoing basin is an $n-1$-disc $\mathcal{N}^{n-1}$ which being of codimension 1, separates $\mathcal{N}$ into two pieces, one containing Π_1 and the other Π_2. The ingoing basin is a 1-disc $\mathcal{N}^1$ which itself is divided in two by $\mathcal{N}^{n-1}$ so that it stretches into Π_1 and Π_2 across the critical point

In the associated Morse geometry (N, g), where $N = \mathcal{N} - p$, define $\mathcal{P} \equiv I^-(N^1, N)$ and $\mathcal{F} \equiv I^+(N^{n-1}, N)$ as before. $\mathcal{P}$ splits into $\mathcal{P}_1 \amalg \mathcal{P}_2$, where $\mathcal{P}_i = \mathcal{P} \cap \Pi_i$. Similarly $\partial\mathcal{F}$ has two components $\partial\mathcal{F}_1$ and $\partial\mathcal{F}_2$, one within each of the halves in which N^{n-1} divides N. Choose $\partial\mathcal{F}_1$ to be in the half which also contains $\mathcal{P}_1$.

Take points $s \in \mathcal{P}_2$ and $q \in \partial\mathcal{F}_1$ close enough to p so that a timelike curve from s to q, if there is one, must be totally contained in N. Note that claims 3 and 4 guarantee the existence of such s and q in N. Then $s \notin I^-(q)$, since if there were a timelike curve from s to q it would have to cross the separating disc N^{n-1} and we would conclude that $q \in \mathcal{F}$, which is a contradiction, since $\mathcal{F}$ is open. Moreover, since $\mathcal{F}$ is a future set [14], any point $q \in \partial\mathcal{F}$ must satisfy $I^+(q, N) \subset \mathcal{F}$, which combined with claim 2 implies $\mathcal{P} \subset \downarrow_N \mathcal{F} \subset \downarrow_N I^+(q, N)$. Clearly, $\downarrow_N I^+(q, N) \subset \downarrow I^+(q, N)$, and since any $q' \in I^+(q)$ is in the chronological future of some $q'' \in I^+(q, N)$, we also have $\downarrow I^+(q, N) = \downarrow I^+(q)$. It follows that $\mathcal{P} \subset \downarrow I^+(q)$ and therefore $s \in \downarrow I^+(q) - I^-(q)$. $\square$

Having dealt with this most general proof of causal discontinuity for all index 1 and $n-1$ neighbourhood Morse geometries, we would like to construct a similar proof of causal continuity for index $\neq 1, n-1$. However, such a proof requires more than the topological equivalence of ξ and η, since we need not one family, ξ, but *all* possible families of timelike curves.

However we do have examples of causally continuous index $\neq 1, n-1$ neighbourhood geometries from [4], namely the "Cartesian" ones. We use this result crucially

in the construction of the main proposition. We now proceed to examine cobordisms with no index $1, n-1$ points.

5 Cobordisms with no index 1 or $n-1$ points

A corollary of Lemma 2 is that a cobordism for which every Morse function contains Morse points of index 1 or n–1 supports no causally continuous Morse geometries. We now show that any cobordism which admits Morse functions without index 1 or $n-1$ points supports causally continuous Morse geometries. We will do so by combining the causal continuity of the Cartesian neighbourhood Morse geometries, (Q_δ, g) of type $(\lambda, n-\lambda)$ when $\lambda \neq 1, n-1$ (see [4]), with the "stacking" and "insertion" results proved below. In the proofs we will be using the following two claims.

Let U be an open subset of the spacetime M. We have already seen that $\downarrow_U I^+(y,U) \subset \downarrow I^+(y) \cap U$ for every $y \in U$. For a certain class of subsets U the converse also holds.

Claim 5 *Let (M,g) be an arbitrary spacetime and U an open subset of M such that for every point $x \in U$ we have $I^\pm(x,U) = I^\pm(x) \cap U$. Then for each $y \in U$ we have $\downarrow_U I^+(y,U) = \downarrow I^+(y) \cap U$.*

Proof Suppose x lies in $\downarrow I^+(y) \cap U$. It must have a neighbourhood U_x in $\downarrow I^+(y) \cap U$. For each $y' \in I^+(y,U)$, $U_x \subset I^-(y') \cap U = I^-(y',U)$, so that $U_x \subset \downarrow_U I^+(y,U)$. $\square$

The dual result is proved similarly. Claim 5 clearly holds for any open subset U which is I-convex in M. A related result, important to our study of causal continuity is:

Claim 6 *Let (M,g) be an arbitrary spacetime and U an open subset of M such that for every point $x \in U$, $I^\pm(x,U) = I^\pm(x) \cap U$. If U is causally discontinuous then M is causally discontinuous.*

Proof Suppose that for points x, y in U we have $x \in \downarrow_U I^+(y,U) - I^-(y,U)$. We know that $x \in \downarrow I^+(y)$. And since $I^-(y,U) = I^-(y) \cap U$, we must have $x \notin I^-(y)$. If $\uparrow_U I^-(y,U) \neq I^+(y,U)$ one proceeds similarly to show that $\uparrow I^-(y) \neq I^+(y)$. $\square$

5.1 Stacking cobordisms

Lemma 3 *Consider a Morse geometry (M, g) defined through the Morse function f and a Riemannian metric h. Let $M_1 = f^{-1}([0, b))$ and $M_2 = f^{-1}((a, 1])$, with $a < b$, so that $M = M_1 \cup M_2$. Then (M, g) is causally continuous iff both (M_1, g) and (M_2, g) are causally continuous.*

Proof Clearly M_1 and M_2 are I-convex in M, since any timelike curve γ between two points x and y, in say M_1, must satisfy $f(x) \leq f(\gamma(t)) \leq f(y)$ and hence be contained in M_1. Therefore, if either M_1 or M_2 is causally discontinuous, then by claim 6, M too will be causally discontinuous.

Now, let both M_1 and M_2 be causally continuous. Clearly, for $x \in M_1$, $I^-(x) = I^-(x, M_1)$, so that $\downarrow I^+(x) \subset \downarrow I^+(x, M_1) = I^-(x)$. Similarly, for y in M_2 we have $\uparrow I^-(y) = I^+(y)$. Thus, it remains for us to show that $I^-(y) = \downarrow I^+(y)$ for every $y \in M_2$. By a dual argument, one would find that $I^+(x) = \uparrow I^-(x)$ for every $x \in M_1$.

Let us assume the contrary, i.e., $\exists\, y$ such that $\downarrow I^+(y) - I^-(y) \neq \emptyset$. This means that $\downarrow I^+(y) - \overline{I^-(y)} \neq \emptyset$. Let $x \in \downarrow I^+(y) - \overline{I^-(y)}$. Now, x must belong to $M_1 - M_2 \cap M_1$. Otherwise, from convexity of M_2 and claim 5 we would have $x \in \downarrow_{M_2} I^+(y, M_2) - I^-(y, M_2)$, thus contradicting the causal continuity of (M_2, g).

Next, consider a sequence of points $y_k \to y$ with $y_k \in I^+(y)$. There exists a sequence of timelike curves γ_k from x to y_k. Choose a number d with $a < d < b$ so that $\Sigma \equiv f^{-1}(d) \subset M_1 \cap M_2$ is a regular level surface of f and therefore a closed $n-1$ manifold. Each γ_k intersects Σ at a point z_k. Since Σ is compact, a subsequence of the z_k converges to a point $z \in \Sigma$. For every k such that z_k doesn't belong to this subsequence throw away z_k, γ_k and y_k and relabel the remaining subsequences as z_k, γ_k and y_k. Clearly $z \in \overline{I^+(x)}$ which, together with causal continuity of M_1 implies $x \in \overline{I^-(z)}$.

Now, for any $z' \in I^-(z)$ there is a tail of the sequence z_k contained in $I^+(z')$ so that $y \in \overline{I^+(z')}$. Therefore, by causal continuity of M_2 we have $z' \in \overline{I^-(y)}$. Since $I^-(z)$ contains points arbitrarily close to z, we see that $z \in \overline{I^-(y)}$. Transitivity of $\overline{I^-}$ then implies that $x \in \overline{I^-(y)}$, a contradiction. □

Any Morse geometry defined through a Morse function f which has just one critical point per critical value can be decomposed into elementary cobordisms, stacked together as M_1 and M_2 in figure 3. By repeatedly applying lemma 3 we obtain

Corollary 1 *Let (M, g) be a Morse geometry associated with a cobordism $\mathcal{M}$ and a Morse function $f : \mathcal{M} \to \mathbb{R}$ which has one critical point per critical value. Then (M, g) is causally continuous if and only if each of the elementary Morse geometries into which it decomposes is causally continuous.*

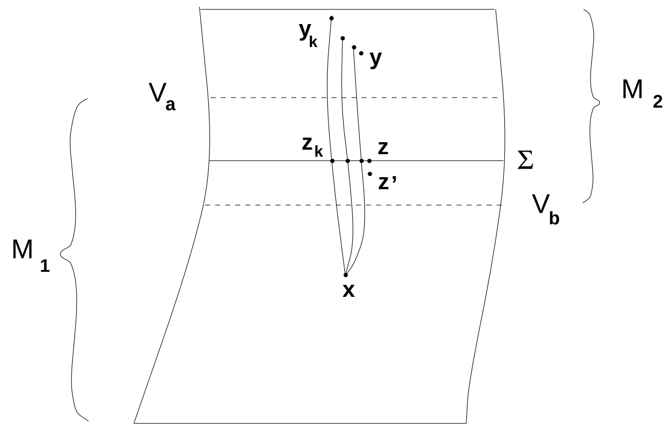

Figure 3: Divide the Morse geometry M in two blocks $M = M_1 \cup M_2$, bounded by level surfaces of the Morse function f. If (M_1, g) and (M_2, g) are causally continuous, so must be M.

5.2 Inserting Morse neighbourhoods

We start by showing that the converse of claim 6 holds in an elementary Morse geometry (E, g), when the open set in question is a neighbourhood of the critical point.

Lemma 4 *Let $\mathcal{E}$ be an elementary cobordism, (E, g) be an associated Morse geometry, and $Q \subset E$ be a neighbourhood of the Morse point such that $I^{\pm}(x, Q) = I^{\pm}(x) \cap Q$ $\forall x \in Q$. If (Q, g) is causally continuous, then (E, g) too is causally continuous.*

Guided by the proof of the stacking lemma, we seek a neighbourhood Q' of p, an "elsewhere region" S and a compact hypersurface Σ which satisfy: (i) (Q', g) and (S, g) are causally continuous, $I^{\pm}(x, Q') = I^{\pm}(x) \cap Q'$ $\forall x \in Q'$ and $I^{\pm}(x, S) = I^{\pm}(x) \cap S$ $\forall x \in S$; (ii) there is an elementary cobordism $E' = f^{-1}([a, b]) \subset E$ with $E' = Q' \cup S$; (iii) the hypersurface Σ is contained in $Q' \cap S$ and separates $Q' - S \cap Q'$ and $S - Q' \cap S$. Then we can proceed as in the proof of lemma 3, with Q' and S playing the role of M_1 and M_2 there.

Proof Let (Q, g) be causally continuous. Let $D_A \subset D_B \subset D_C$ be open n-discs centered at p such that $\overline{D_C} \subset Q$, with $n-1$-sphere boundaries S_A, S_B and S_C, respectively. We know from the proof of claim 3 that there is an α with $0 < \alpha < min(c, 1-c)$ such that along any future-directed causal curve γ with an arc between S_A and S_B, the Morse function must increase by at least α. Let $\Pi = f^{-1}([0, c))$ and $\Phi = f^{-1}((c, 1])$ be the subcritical and supercritical regions in E, respectively. Thus if $x \in D_A \cap \Pi$ and $y \in I^{-}(x) \cap (E - D_B)$, then $f(y) < c - \alpha$ and similarly if $x \in D_A \cap \Phi$

and $y \in I^+(x) \cap (E - D_B)$ then $f(y) > c + \alpha$. Now consider $a = c - \alpha$ and $b = c + \alpha$, so that $E' = f^{-1}([a, b])$ is the associated thinner elementary Morse geometry contained in E.

Define $Q' \equiv Q \cap E'$ and $S \equiv D(V_c, E')$, the domain of dependence of the critical surface in (E', g). Moreover, let $\Sigma \equiv S_C \cap E'$. We now demonstrate that Q', S and Σ satisfy the conditions (i), (ii) and (iii) mentioned above, so that the proof reduces to the proof of the stacking lemma:

(i) By construction, the set $Q' \subset Q$ satisfies $I^{\pm}(x, Q') = I^{\pm}(x) \cap Q'$ $\forall\, x \in Q'$. It follows from claim 6 that (Q', g) is causally continuous. The set S is convex and causally continuous, since V_c is a closed achronal subset of E (see section 2).

(ii) To check that $E' = Q' \cup S$ we just need to verify that any point $x \in E' - D_B \cap E'$ is in S, since $D_B \cap E' \subset Q'$. We assume that $x \in \Phi$, the case with $x \in \Pi$ follows from dual arguments. So suppose there is a past-inextendible causal curve through x which does not intersect V_c. Then, since $J^-(x) \cap D_A = \emptyset$, it must be confined in the region $f^{-1}([c, f(x)]) - \left(D_A \cap f^{-1}([c, f(x)])\right)$, which is homeomorphic to $V_1 \times [0, 1]$, where V_1 is the future boundary of E, and therefore compact. This is impossible, since (E, g) is strongly causal.

(iii) Finally, it is clear from the construction that $\Sigma = S_C \cap E'$ is homeomorphic to $S^{\lambda-1} \times S^{n-\lambda-1} \times [0, 1]$, and therefore compact. Moreover Σ separates $Q' - S \cap Q'$ from $S - Q' \cap S$.

We now demonstrate that (E', g) is causally continuous. Causal continuity of (E, g) will then follow, since E can be expressed as $M_1 \cup E' \cup M_2$, with M_1 and M_2 being causally continuous product cobordisms, and then the stacking lemma applies.

We assume the contrary, i.e. that (E', g) is causally discontinuous. Thus, there exist points x and y in $\mathrm{Int}\,(E')$ satisfying $x \in \downarrow I^+(y) - \overline{I^-(y)}$. Because $I^{\pm}(z, Q') = I^{\pm}(z) \cap Q'$ $\forall\, z \in Q$, claim 5 tells us that $\downarrow_{Q'} I^+(y, Q') = \downarrow I^+(y) \cap Q'$. Thus, if both x and y belong to Q' this would mean that (Q', g) is causally discontinuous, which is a contradiction. Similarly, the causal continuity of S means that x and y cannot both be in S. Thus, $x \in Q' - S \cap S$ and $y \in S - Q' \cap S$ or vice-versa. Let us consider the first possibility, i.e., $x \in Q' - S \cap Q'$. The other possibility is covered by swapping S and Q' in the argument that follows.

Consider a sequence of points $y_k \to y$ with $y_k \in I^+(y)$. There exists a sequence of timelike curves γ_k from x to y_k, all of which intersect Σ, since by (iii) any curve from a point x in $Q' - S \cap Q'$ to a point y in $S - Q' \cap S$ must cross Σ at least once. Let z_k be the first point at which each γ_k intersects Σ. A subsequence of the z_k converges to a point $z \in \Sigma$. From here one proceeds exactly as in the proof of lemma 3, namely one uses the causal continuity of S and Q' to deduce $x \in \overline{I^-(y)}$, a contradiction. □

This result can be easily extended to the case where there are several Morse points in the same critical surface. If each critical point p_j has a neighbourhood Q_j such that (Q_j, g) is causally continuous and $I^{\pm}(x, Q_j) = I^{\pm}(x) \cap Q_j \;\forall\, x \in Q_j$, then (M, g) too will be causally continuous. Therefore, to establish that every Morse geometry (M, g) without index 1 or $n-1$ points is causally continuous it would suffice to find that around a critical point of index $\lambda \neq 1, n-1$ there is a neighbourhood Q such that (Q, g) is causally continuous and Q is I-convex in M.

The following claim guarantees that any neighbourhood of a critical point contains neighbourhoods which are I-convex relative to the whole Morse spacetime.

Claim 7 *Let (M, g) be a Morse geometry and $\mathcal{D}_\epsilon$ be a round neighbourhood of a critical point $p \in \mathcal{M}$, then there exists a punctured neighbourhood $Q \subset \mathcal{D}_\epsilon$ of p which is I-convex with respect to (M, g).*

Proof Let h be the Riemannian metric and f the Morse function from which g is constructed. From lemma 1 we know there is a homeomorphism $\Psi : \mathcal{D}_{\epsilon'} \to \mathcal{N}$ between neighbourhoods $\mathcal{D}_{\epsilon'}$, $\mathcal{N}$ of p contained in $\mathcal{D}_\epsilon$, so that we know the topology of the ingoing and outgoing basins of ξ. Let $c = f(p)$ be the critical value and let the numbers a,b, with $a < c < b$, be such that the sets $A_\eta \equiv V_a \cap D^\lambda_{\epsilon'}$ and $B_\eta \equiv V_b \cap D^{n-\lambda}_{\epsilon'}$ (where $V_a \equiv f^{-1}(a)$, *etc.*) are respectively a $\lambda - 1$ sphere and an $n-\lambda-1$ sphere. Their images $A_\xi = \Psi(A_\eta) \subset \mathcal{N}^\lambda$ and $B_\xi = \Psi(B_\eta) \subset \mathcal{N}^{n-\lambda}$ are also spheres. Define the set $Q = I^+(A_\xi) \cap I^-(B_\xi)$ in M. Then Q is I-convex in M.

That Q is a neighbourhood of p can be immediately seen from the topological equivalence between the η and the ξ flows in $\mathcal{N}$. We can ensure that Q is contained in D_ϵ, by choosing the numbers a and b close enough to c. To see this, let U be a punctured neighbourhood of p with $\overline{U} \subset D_\epsilon$. From claim 3 we know that there is a number α_U such that for any $x \in U$ and $z \notin D_\epsilon$ connected through a timelike curve we must have $|f(x) - f(z)| > \alpha_U$. Let $U' \subset U$ be small enough so that for any $x \in U'$ we have $|f(x) - c| < \alpha_U$. We can assume that $N = \mathcal{N} - p \subset U'$. Define then $a = c - \alpha$ and $b = c + \alpha$ for some small $\alpha < \alpha_U$ so that A_η, B_η are spheres in N. Then A_ξ and B_ξ are contained in U' and therefore $I^+(A_\xi) \cap I^-(B_\xi)$ is contained in D_ϵ. Otherwise, if there were points $z \notin D_\epsilon$, $x \in A_\xi$ and $y \in B_\xi$ with $z \in I^+(x) \cap I^-(y)$, we would get the contradiction $c < f(x) + \alpha < f(z) < f(y) - \alpha < c$. $\square$

We have established this result here because it may be needed in the future, to complete the proof of the Borde-Sorkin conjecture. We do not need it for our present purposes, since directly from lemma 4 we obtain:

Corollary 2 *Let $\mathcal{E}$ be an elementary cobordism and $f : \mathcal{E} \to [0, 1]$ a Morse function with a single Morse point p of index $\lambda \neq 1, n-1$. Let g be a Morse metric constructed*

from f and a Riemannian metric, h, which is Cartesian in a neighbourhood of the Morse point. Then (E,g) is causally continuous.

Indeed, we know from previous work [4] that the neighbourhood in which h is Cartesian contains a causally continuous neighbourhood Morse geometry (Q_δ, g) where Q_δ is I-convex. In fact, Q_δ has precisely the form $Q \cap E'$ with $Q = I^+(A_\eta) \cap I^-(B_\eta)$, as in claim 7, and E' an open elementary cobordism in E.

5.3 Causal continuity when $\lambda \neq 1, n-1$

Finally, we show that if the cobordism $\mathcal{M}$ has a Morse function f_0 without index 1 or $n-1$ critical points, then there exist causally continuous Morse spacetimes associated with $\mathcal{M}$. We do this by constructing a Riemannian metric h_0 such that near the critical points $\{p_k\}$, h_0 is Cartesian flat in precisely the same coordinates in which f_0 takes the form (2). We start by covering $\mathcal{M}$ with a finite atlas $\{U_\alpha\}$ such that the chart U_k with $p_k \in U_k$ contains a round neighbourhood D_k of p_k where the Morse function f_0 takes its canonical form and such that D_k intersects none of the other charts. We construct the Riemannian metric h_0 using an associated partition of unity $\{\theta_\alpha\}$ to patch together local metrics:

$$h_{0\mu\nu}(x) = \sum_\alpha \theta_\alpha(x) h^\alpha_{\mu\nu}(x) \tag{9}$$

When $\alpha = k$ corresponds to a chart containing critical point p_k, $h^k_{\mu\nu} = \delta_{\mu\nu}$. The other $h^\alpha_{\mu\nu}$ are arbitrary. With such an h_0, we can apply corollary 2 and corollary 1 and conclude that (M, g_0) is causally continuous.

6 Conclusions

Our results suffice to establish the following proposition which, if weaker than the Borde-Sorkin conjecture, suggests the same selection rule for cobordisms in the gravitational Sum-Over-Histories.

Proposition 1 *Given a compact cobordism $\mathcal{M}$ and a Morse function $f : \mathcal{M} \rightarrow \mathrm{I\!R}$ then: (i) if f has no critical points with Morse index 1 or $n-1$, there exist Morse spacetimes (M,g) associated with $\mathcal{M}$ which are causally continuous; (ii) if f has critical points of index 1 or $n-1$, every Morse geometry (M,g) defined through f is causally discontinuous.*

Another useful way of stating this result is,

Corollary 3 *If $\mathcal{M}$ is a compact manifold admitting only Morse functions containing critical points of index 1 or n-1 then it can support only causally discontinuous Morse spacetimes. If $\mathcal{M}$ admits a Morse function which possesses no critical points of index 1 or n-1, then it can support causally continuous Morse spacetimes.*

In [2, 3], we examined certain topologies corresponding to important physical processes, namely, the pair production of black holes, Kaluza Klein monopoles and geons. We found that while the black hole and Kaluza Klein cases admit Morse functions which have no index 1 or $n-1$ points, this is not true for the pair production of irreducible geons. Corollary 3 then tells us that the black hole and monopole pair-production topologies in fact admit causally continuous Morse histories, while the geon pair-production topologies do not. In [2] we moreover stated that there always exists a topological transition between any two 3 manifolds which admits a Morse function with no index $1, n-1$ point, which means therefore that such a transition always admits a causally continuous Morse history.

It of course remains to be seen if the Sorkin conjecture which relates causal continuity of a distinguishing spacetime to the non-singular propagation of quantum fields on that background, can be verified in higher dimensions. We leave this for future investigations.

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